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Intrinsic square functions on the weighted Morrey spaces

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ABSTRACT

In this paper, we will study the boundedness properties of intrinsic square functions including the Lusin area integral, Littlewood–Paley g-function and g^*_λ-function on the weighted Morrey spaces $L^{p,\kappa}(w)$ for $1 < p < \infty$ and $0 < \kappa < 1$. The corresponding commutators generated by $BMO(\mathbb{R}^n)$ functions and intrinsic square functions are also discussed.

1. Introduction and main results

Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$. The classical square function (Lusin area integral) is a familiar object. If $u(x,t) = P_t * f(x)$ is the Poisson integral of f, where $P_t(x) = c_n \frac{t}{(t^2+|x|^2)^{(n+1)/2}}$ denotes the Poisson kernel in $\mathbb{R}^{n+1}_+$, then we define the classical square function (Lusin area integral) $S(f)$ by

$$S(f)(x) = \left(\iint_{\Gamma(x)} |\nabla u(y,t)|^2 t^{1-n}\, dydt\right)^{1/2},$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x) = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\right\}$$

and

$$|\nabla u(y,t)| = \left|\frac{\partial u}{\partial t}\right|^2 + \sum_{j=1}^n \left|\frac{\partial u}{\partial y_j}\right|^2.$$

We can similarly define a cone of aperture β for any $\beta > 0$:

$$\Gamma_\beta(x) = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\right\},$$

and corresponding square function

$$S_\beta(f)(x) = \left(\iint_{\Gamma_\beta(x)} |\nabla u(y,t)|^2 t^{1-n}\, dydt\right)^{1/2}.$$

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The Littlewood–Paley g-function (could be viewed as a "zero-aperture" version of $S(f)$) and the g_λ^*-function (could be viewed as an "infinite aperture" version of $S(f)$) are defined respectively by

$$g(f)(x)=\left(\int_0^\infty |\nabla u(x,t)|^2\, t\, dt\right)^{1/2}$$

and

$$g_\lambda^*(f)(x)=\left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}|\nabla u(y,t)|^2\, t^{1-n}\, dydt\right)^{1/2}.$$

The modern (real-variable) variant of $S_\beta(f)$ can be defined in the following way. Let $\psi\in C^\infty(\mathbb{R}^n)$ be real, radial, have support contained in $\{x:|x|\le 1\}$, and $\int_{\mathbb{R}^n}\psi(x)\,dx=0$. The continuous square function $S_{\psi,\beta}(f)$ is defined by

$$S_{\psi,\beta}(f)(x)=\left(\iint_{\Gamma_\beta(x)}|f*\psi_t(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}.$$

In 2007, Wilson [1] introduced a new square function called intrinsic square function which is universal in a sense (see also [2]). This function is independent of any particular kernel ψ, and it dominates pointwise all the above-defined square functions. On the other hand, it is not essentially larger than any particular $S_{\psi,\beta}(f)$. For $0<\alpha\le 1$, let $\mathcal{C}_\alpha$ be the family of functions φ defined on $\mathbb{R}^n$ such that φ has support containing in $\{x\in\mathbb{R}^n:|x|\le 1\}$, $\int_{\mathbb{R}^n}\varphi(x)\,dx=0$, and, for all $x,x'\in\mathbb{R}^n$,

$$|\varphi(x)-\varphi(x')|\le |x-x'|^\alpha.$$

For $(y,t)\in\mathbb{R}_+^{n+1}$ and $f\in L^1_{loc}(\mathbb{R}^n)$, we set

$$A_\alpha(f)(y,t)=\sup_{\varphi\in\mathcal{C}_\alpha}|f*\varphi_t(y)|=\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f(z)\,dz\right|.$$

Then we define the intrinsic square function of f (of order α) by the formula

$$\mathcal{S}_\alpha(f)(x)=\left(\iint_{\Gamma(x)}\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\right)^{1/2}.$$

We can also define varying-aperture versions of $\mathcal{S}_\alpha(f)$ by the formula

$$\mathcal{S}_{\alpha,\beta}(f)(x)=\left(\iint_{\Gamma_\beta(x)}\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\right)^{1/2}.$$

The intrinsic Littlewood–Paley g-function and the intrinsic g_λ^*-function will be defined respectively by

$$g_\alpha(f)(x)=\left(\int_0^\infty\Big(A_\alpha(f)(x,t)\Big)^2\frac{dt}{t}\right)^{1/2}$$

and

$$g_{\lambda,\alpha}^*(f)(x)=\left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\right)^{1/2}.$$

In [2], Wilson proved the following result.

Theorem A. *Let $0<\alpha\le 1$, $1<p<\infty$ and $w\in A_p$ (Muckenhoupt weight class). Then there exists a constant $C>0$ independent of f such that*

$$\|\mathcal{S}_\alpha(f)\|_{L^p_w}\le C\|f\|_{L^p_w}.$$

Moreover, in [3], Lerner showed sharp L^p_w norm inequalities for the intrinsic square functions in terms of the A_p characteristic constant of w for all $1<p<\infty$. As for the boundedness of intrinsic square functions on the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for $n/(n+\alpha)\le p\le 1$, we refer the readers to [4–6].

Let b be a locally integrable function on $\mathbb{R}^n$, in this paper, we will also consider the commutators generated by b and intrinsic square functions, which are defined respectively by the following expressions

$$\big[b,\mathcal{S}_\alpha\big](f)(x)=\left(\iint_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(x)-b(z)\big]\varphi_t(y-z)f(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

$$\big[b,g_\alpha\big](f)(x)=\left(\int_0^\infty\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(x)-b(y)\big]\varphi_t(x-y)f(y)\,dy\right|^2\frac{dt}{t}\right)^{1/2},$$

and

$$\left[b, g^*_{\lambda,\alpha}\right](f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} \left|\int_{\mathbb{R}^n} \left[b(x)-b(z)\right]\varphi_t(y-z)f(z)\,dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}.$$

The classical Morrey spaces $\mathcal{L}^{p,\lambda}$ were first introduced by Morrey in [7] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the readers to [8–10]. For the properties and applications of classical Morrey spaces, see [11–13] and the references therein.

In 2009, Komori and Shirai [14] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ which could be viewed as an extension of weighted Lebesgue spaces, and studied the boundedness of the above classical operators on these weighted spaces. Recently, in [15–17], we have established the continuity properties of some other operators on the weighted Morrey spaces $L^{p,\kappa}(w)$. In the meanwhile, it should be pointed out that in [18], weighted Morrey spaces of different types were defined and the boundedness of some fractional integral operators in these spaces was also given.

The purpose of this paper is to discuss the boundedness properties of intrinsic square functions and their commutators on the weighted Morrey spaces $L^{p,\kappa}(w)$ for all $1 < p < \infty$ and $0 < \kappa < 1$. Our main results in the paper are formulated as follows.

Theorem 1.1. *Let $0 < \alpha \le 1$, $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Then there is a constant $C > 0$ independent of f such that*

$$\|\mathcal{S}_\alpha(f)\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

Theorem 1.2. *Let $0 < \alpha \le 1$, $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Suppose that $b \in BMO(\mathbb{R}^n)$, then there is a constant $C > 0$ independent of f such that*

$$\left\|\left[b, \mathcal{S}_\alpha\right](f)\right\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

Theorem 1.3. *Let $0 < \alpha \le 1$, $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. If $\lambda > \max\{p, 3\}$, then there is a constant $C > 0$ independent of f such that*

$$\left\|g^*_{\lambda,\alpha}(f)\right\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

Theorem 1.4. *Let $0 < \alpha \le 1$, $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. If $b \in BMO(\mathbb{R}^n)$ and $\lambda > \max\{p, 3\}$, then there is a constant $C > 0$ independent of f such that*

$$\left\|\left[b, g^*_{\lambda,\alpha}\right](f)\right\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

In [1], Wilson also showed that for any $0 < \alpha \le 1$, the functions $\mathcal{S}_\alpha(f)(x)$ and $g_\alpha(f)(x)$ are pointwise comparable. Thus, as a direct consequence of Theorems 1.1 and 1.2, we obtain the following.

Corollary 1.5. *Let $0 < \alpha \le 1$, $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Then there is a constant $C > 0$ independent of f such that*

$$\|g_\alpha(f)\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

Corollary 1.6. *Let $0 < \alpha \le 1$, $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Suppose that $b \in BMO(\mathbb{R}^n)$, then there is a constant $C > 0$ independent of f such that*

$$\left\|\left[b, g_\alpha\right](f)\right\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

2. Notations and definitions

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy–Littlewood maximal functions in [19]. A weight w is a nonnegative, locally integrable function on $\mathbb{R}^n$, $B = B(x_0, r_B)$ denotes the ball with center x_0 and radius r_B. Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B. For a given weight function w and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and the weighted measure of E by $w(E)$, where $w(E) = \int_E w(x)\,dx$. We say that w is in the Muckenhoupt class A_p with $1 < p < \infty$, if there exists a constant $C > 0$ such that for every ball $B \subseteq \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C.$$

The smallest constant C such that the above inequality holds is called the A_p characteristic constant of w and denoted by $[w]_{A_p}$. A weight function w is said to belong to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds for every ball $B \subseteq \mathbb{R}^n$.

$$\left(\frac{1}{|B|}\int_B w(x)^r\,dx\right)^{1/r} \le C\left(\frac{1}{|B|}\int_B w(x)\,dx\right).$$

We state the following results that we will use frequently in the sequel.

Lemma 2.1 ([20]). *Let $w \in A_p$ with $1 < p < \infty$. Then, for any ball B, there exists an absolute constant $C > 0$ such that*

$$w(2B) \le C\,w(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \le C \cdot \lambda^{np} w(B),$$

where C does not depend on B or λ.

Lemma 2.2 ([21]). *Let $w \in RH_r$ with $r > 1$. Then there exists a constant $C > 0$ such that*

$$\frac{w(E)}{w(B)} \le C\left(\frac{|E|}{|B|}\right)^{(r-1)/r}$$

for any measurable subset E of a ball B.

Given a weight function w on $\mathbb{R}^n$, for $1 < p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the space of all functions satisfying

$$\|f\|_{L^p_w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$

A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|\,dx < \infty,$$

where b_B stands for the average of b on B, i.e., $b_B = \frac{1}{|B|}\int_B b(y)\,dy$ and the supremum is taken over all balls B in $\mathbb{R}^n$.

Theorem 2.3 ([22,23]). *Assume that $b \in BMO(\mathbb{R}^n)$. Then for any $1 \le p < \infty$, we have*

$$\sup_B \left(\frac{1}{|B|}\int_B |b(x) - b_B|^p\,dx\right)^{1/p} \le C\|b\|_*.$$

Definition 2.4 ([14]). Let $1 \le p < \infty$, $0 < \kappa < 1$ and w be a weight function. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(w) = \left\{f \in L^p_{loc}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\right\},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa}\int_B |f(x)|^p w(x)\,dx\right)^{1/p}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. Moreover, we will denote the conjugate exponent of $p > 1$ by $p' = p/(p-1)$.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ and decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$, χ_{2B} denotes the characteristic function of $2B$. Since $\mathcal{S}_\alpha (0 < \alpha \le 1)$ is a sublinear operator, then we have

$$\begin{aligned}
&\frac{1}{w(B)^{\kappa/p}}\left(\int_B |\mathcal{S}_\alpha(f)(x)|^p w(x)\,dx\right)^{1/p} \\
&\le \frac{1}{w(B)^{\kappa/p}}\left(\int_B |\mathcal{S}_\alpha(f_1)(x)|^p w(x)\,dx\right)^{1/p} + \frac{1}{w(B)^{\kappa/p}}\left(\int_B |\mathcal{S}_\alpha(f_2)(x)|^p w(x)\,dx\right)^{1/p} \\
&= I_1 + I_2.
\end{aligned}$$

Theorem A and Lemma 2.1 imply

$$\begin{aligned}
I_1 &\le C\cdot\frac{1}{w(B)^{\kappa/p}}\left(\int_{2B} |f(x)|^p w(x)\,dx\right)^{1/p} \\
&\le C\|f\|_{L^{p,\kappa}(w)}\cdot\frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\
&\le C\|f\|_{L^{p,\kappa}(w)}.
\end{aligned}$$

We now turn to estimate the other term I_2. For any $\varphi \in \mathcal{C}_\alpha$, $0 < \alpha \le 1$ and $(y,t) \in \Gamma(x)$, we have

$$\begin{aligned}
|f_2 * \varphi_t(y)| &= \left|\int_{(2B)^c} \varphi_t(y-z) f(z)\,dz\right| \\
&\le C\cdot t^{-n}\int_{(2B)^c\cap\{z:|y-z|\le t\}} |f(z)|\,dz \\
&\le C\cdot t^{-n}\sum_{k=1}^\infty \int_{(2^{k+1}B\setminus 2^kB)\cap\{z:|y-z|\le t\}} |f(z)|\,dz.
\end{aligned} \tag{1}$$

For any $x \in B$, $(y,t) \in \Gamma(x)$ and $z \in \left(2^{k+1}B \setminus 2^kB\right) \cap B(y,t)$, then by a direct computation, we can easily see that

$$2t \ge |x-y| + |y-z| \ge |x-z| \ge |z-x_0| - |x-x_0| \ge 2^{k-1}r_B.$$

Thus, by using the above inequality (1) and Minkowski's integral inequality, we deduce that

$$\begin{aligned}
|\mathcal{S}_\alpha(f_2)(x)| &= \left(\iint_{\Gamma(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |f_2 * \varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \\
&\le C\left(\int_{2^{k-2}r_B}^\infty \int_{|x-y|<t} \left|t^{-n}\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} |f(z)|\,dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \\
&\le C\left(\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} |f(z)|\,dz\right)\left(\int_{2^{k-2}r_B}^\infty \frac{dt}{t^{2n+1}}\right)^{1/2} \\
&\le C\sum_{k=1}^\infty \frac{1}{|2^{k+1}B|}\int_{2^{k+1}B\setminus 2^kB} |f(z)|\,dz.
\end{aligned}$$

It follows from Hölder's inequality and the A_p condition that

$$\begin{aligned}
\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} |f(z)|\,dz &\le \frac{1}{|2^{k+1}B|}\left(\int_{2^{k+1}B} |f(z)|^p w(z)\,dz\right)^{1/p}\left(\int_{2^{k+1}B} w(z)^{-p'/p}\,dz\right)^{1/p'} \\
&\le C\|f\|_{L^{p,\kappa}(w)}\cdot w\left(2^{k+1}B\right)^{(\kappa-1)/p}.
\end{aligned} \tag{2}$$

Hence

$$I_2 \le C\|f\|_{L^{p,\kappa}(w)}\sum_{k=1}^\infty \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}}.$$

Since $w \in A_p$ with $1 < p < \infty$, then there exists a number $r > 1$ such that $w \in RH_r$. Consequently, by using Lemma 2.2, we can get

$$\frac{w(B)}{w(2^{k+1}B)} \le C\left(\frac{|B|}{|2^{k+1}B|}\right)^{(r-1)/r}. \tag{3}$$

Therefore

$$\begin{aligned} I_2 &\le C\|f\|_{L^{p,\kappa}(w)}\sum_{k=1}^{\infty}\left(\frac{1}{2^{kn}}\right)^{(1-\kappa)(r-1)/pr} \\ &\le C\|f\|_{L^{p,\kappa}(w)}, \end{aligned}$$

where the last series is convergent since $(1-\kappa)(r-1)/pr > 0$. Combining the above estimates for I_1 and I_2 and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we complete the proof of Theorem 1.1. $\square$

Given a real-valued function $b \in BMO(\mathbb{R}^n)$, we shall follow the idea developed in [24,25] and denote $F(\xi) = e^{\xi[b(x)-b(z)]}$, $\xi \in \mathbb{C}$. Then by the analyticity of $F(\xi)$ on $\mathbb{C}$ and the Cauchy integral formula, we get

$$\begin{aligned} b(x)-b(z) = F'(0) &= \frac{1}{2\pi i}\int_{|\xi|=1}\frac{F(\xi)}{\xi^2}\,d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}[b(x)-b(z)]}e^{-i\theta}\,d\theta. \end{aligned}$$

Thus, for any $\varphi \in \mathcal{C}_\alpha$, $0 < \alpha \le 1$, we obtain

$$\begin{aligned} \left|\int_{\mathbb{R}^n}\big[b(x)-b(z)\big]\varphi_t(y-z)f(z)\,dz\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\left(\int_{\mathbb{R}^n}\varphi_t(y-z)e^{-e^{i\theta}b(z)}f(z)\,dz\right)e^{e^{i\theta}b(x)}e^{-i\theta}\,d\theta\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)e^{-e^{i\theta}b(z)}f(z)\,dz\right|e^{\cos\theta\cdot b(x)}\,d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi}A_\alpha\big(e^{-e^{i\theta}b}\cdot f\big)(y,t)\cdot e^{\cos\theta\cdot b(x)}\,d\theta. \end{aligned}$$

So we have

$$\begin{aligned} \big|\big[b,\mathcal{S}_\alpha\big](f)(x)\big| &\le \frac{1}{2\pi}\int_0^{2\pi}\mathcal{S}_\alpha\big(e^{-e^{i\theta}b}\cdot f\big)(x)\cdot e^{\cos\theta\cdot b(x)}\,d\theta, \\ \big|\big[b,g^*_{\lambda,\alpha}\big](f)(x)\big| &\le \frac{1}{2\pi}\int_0^{2\pi}g^*_{\lambda,\alpha}\big(e^{-e^{i\theta}b}\cdot f\big)(x)\cdot e^{\cos\theta\cdot b(x)}\,d\theta. \end{aligned}$$

Then, by using the same arguments as in [25], we can also show the following.

Theorem 3.1. *Let $0 < \alpha \le 1$, $1 < p < \infty$ and $w \in A_p$. Then the commutators $\big[b,\mathcal{S}_\alpha\big]$ and $\big[b,g^*_{\lambda,\alpha}\big]$ are all bounded from $L^p_w(\mathbb{R}^n)$ into itself whenever $b \in BMO(\mathbb{R}^n)$.*

Proof of Theorem 1.2. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$. Then we can write

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}}\left(\int_B\big|\big[b,\mathcal{S}_\alpha\big](f)(x)\big|^p w(x)\,dx\right)^{1/p} \\ &\le \frac{1}{w(B)^{\kappa/p}}\left(\int_B\big|\big[b,\mathcal{S}_\alpha\big](f_1)(x)\big|^p w(x)\,dx\right)^{1/p} + \frac{1}{w(B)^{\kappa/p}}\left(\int_B\big|\big[b,\mathcal{S}_\alpha\big](f_2)(x)\big|^p w(x)\,dx\right)^{1/p} \\ &= J_1 + J_2. \end{aligned}$$

Applying Theorem 3.1 and Lemma 2.1, we thus obtain

$$\begin{aligned} J_1 &\le C\cdot\frac{1}{w(B)^{\kappa/p}}\left(\int_{2B}|f(x)|^p w(x)\,dx\right)^{1/p} \\ &\le C\|f\|_{L^{p,\kappa}(w)}\cdot\frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\ &\le C\|f\|_{L^{p,\kappa}(w)}. \end{aligned} \tag{4}$$

We now turn to deal with the term J_2. For any given $x \in B$ and $(y, t) \in \Gamma(x)$, we have

$$\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(x)-b(z)\big]\varphi_t(y-z)f_2(z)\,dz\right| \le |b(x)-b_B|\cdot\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f_2(z)\,dz\right| + \sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(z)-b_B\big]\varphi_t(y-z)f_2(z)\,dz\right|.$$

Hence

$$\begin{aligned}\left|\big[b,\mathcal{S}_\alpha\big](f_2)(x)\right| &\le |b(x)-b_B|\cdot S_\alpha(f_2)(x) + \left(\iint_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(z)-b_B\big]\varphi_t(y-z)f_2(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &= \mathrm{I}+\mathrm{II}.\end{aligned}$$

In the proof of Theorem 1.1, we have already proved that for any $x \in B$,

$$|S_\alpha(f_2)(x)| \le C\|f\|_{L^{p,\kappa}(w)}\cdot\sum_{k=1}^\infty w\big(2^{k+1}B\big)^{(\kappa-1)/p}.$$

Consequently

$$\begin{aligned}\frac{1}{w(B)^{\kappa/p}}\left(\int_B \mathrm{I}^p\,w(x)\,dx\right)^{1/p} &\le C\|f\|_{L^{p,\kappa}(w)}\frac{1}{w(B)^{\kappa/p}}\cdot\sum_{k=1}^\infty w\big(2^{k+1}B\big)^{(\kappa-1)/p}\cdot\left(\int_B|b(x)-b_B|^p w(x)\,dx\right)^{1/p}\\ &= C\|f\|_{L^{p,\kappa}(w)}\sum_{k=1}^\infty\frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}}\cdot\left(\frac{1}{w(B)}\int_B|b(x)-b_B|^p w(x)\,dx\right)^{1/p}.\end{aligned}$$

Using the same arguments as that of Theorem 1.1, we can see that the above summation is bounded by a constant. Hence

$$\frac{1}{w(B)^{\kappa/p}}\left(\int_B \mathrm{I}^p\,w(x)\,dx\right)^{1/p} \le C\|f\|_{L^{p,\kappa}(w)}\left(\frac{1}{w(B)}\int_B|b(x)-b_B|^p w(x)\,dx\right)^{1/p}.$$

Since $w \in A_p$, as before, we know that there exists a number $r > 1$ such that $w \in RH_r$. Thus by Hölder's inequality and Theorem 2.3, we deduce

$$\begin{aligned}\left(\frac{1}{w(B)}\int_B|b(x)-b_B|^p w(x)\,dx\right)^{1/p} &\le \frac{1}{w(B)^{1/p}}\left(\int_B|b(x)-b_B|^{pr'}\,dx\right)^{1/(pr')}\left(\int_B w(x)^r\,dx\right)^{1/(pr)}\\ &\le C\cdot\left(\frac{1}{|B|}\int_B|b(x)-b_B|^{pr'}\,dx\right)^{1/(pr')}\\ &\le C\|b\|_*. \qquad (5)\end{aligned}$$

So we have

$$\frac{1}{w(B)^{\kappa/p}}\left(\int_B \mathrm{I}^p\,w(x)\,dx\right)^{1/p} \le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}. \qquad (6)$$

On the other hand

$$\begin{aligned}\mathrm{II} &= \left(\iint_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{(2B)^c}\big[b(z)-b_B\big]\varphi_t(y-z)f(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &\le C\left(\iint_{\Gamma(x)}\left|t^{-n}\sum_{k=1}^\infty\int_{(2^{k+1}B\backslash 2^kB)\cap\{z:|y-z|\le t\}}|b(z)-b_B||f(z)|\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &\le C\left(\iint_{\Gamma(x)}\left|t^{-n}\sum_{k=1}^\infty\int_{(2^{k+1}B\backslash 2^kB)\cap\{z:|y-z|\le t\}}|b(z)-b_{2^{k+1}B}||f(z)|\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &\quad+C\left(\iint_{\Gamma(x)}\left|t^{-n}\sum_{k=1}^\infty|b_{2^{k+1}B}-b_B|\cdot\int_{(2^{k+1}B\backslash 2^kB)\cap\{z:|y-z|\le t\}}|f(z)|\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &= \mathrm{III}+\mathrm{IV}.\end{aligned}$$

An application of Hölder's inequality gives us that

$$\begin{aligned}\int_{2^{k+1}B\backslash 2^kB}|b(z)-b_{2^{k+1}B}||f(z)|\,dz &\le \left(\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|^{p'}w(z)^{-p'/p}\,dz\right)^{1/p'}\left(\int_{2^{k+1}B}|f(z)|^p w(z)\,dz\right)^{1/p}\\ &\le C\|f\|_{L^{p,\kappa}(w)}\cdot w\big(2^{k+1}B\big)^{\kappa/p}\left(\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|^{p'}w(z)^{-p'/p}\,dz\right)^{1/p'}. \qquad (7)\end{aligned}$$

If we set $\nu(z)=w(z)^{-p'/p}=w(z)^{1-p'}$, then we have $\nu\in A_{p'}$ because $w\in A_p$ (see [20]). Following along the same lines as in the proof of (5), we can also show

$$\left(\frac{1}{\nu(2^{k+1}B)}\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|^{p'}\nu(z)\,dz\right)^{1/p'}\le C\|b\|_*. \qquad (8)$$

Substituting the above inequality (8) into (7), we thus obtain

$$\begin{aligned}\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}||f(z)|\,dz &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\cdot w\big(2^{k+1}B\big)^{\kappa/p}\nu\big(2^{k+1}B\big)^{1/p'}\\ &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\cdot|2^{k+1}B|w\big(2^{k+1}B\big)^{(\kappa-1)/p}.\end{aligned}$$

In addition, we note that in this case, $t\ge 2^{k-2}r_B$ as in Theorem 1.1. From the above inequality, it follows that

$$\begin{aligned}\mathrm{III} &\le C\left(\int_{2^{k-2}r_B}^\infty\int_{|x-y|<t}\left|t^{-n}\sum_{k=1}^\infty\int_{2^{k+1}B\backslash 2^kB}|b(z)-b_{2^{k+1}B}||f(z)|\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &\le C\left(\sum_{k=1}^\infty\int_{2^{k+1}B\backslash 2^kB}|b(z)-b_{2^{k+1}B}||f(z)|\,dz\right)\left(\int_{2^{k-2}r_B}^\infty\frac{dt}{t^{2n+1}}\right)^{1/2}\\ &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\cdot w\big(2^{k+1}B\big)^{(\kappa-1)/p}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{w(B)^{\kappa/p}}\left(\int_B \mathrm{III}^p\,w(x)\,dx\right)^{1/p} &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{k=1}^\infty\frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}}\\ &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}. \qquad (9)\end{aligned}$$

Now let us deal with the last term IV. Since $b\in BMO(\mathbb{R}^n)$, then a simple calculation shows that

$$|b_{2^{k+1}B}-b_B|\le C\cdot(k+1)\|b\|_*. \qquad (10)$$

It follows from the inequalities (2) and (10) that

$$\begin{aligned}\mathrm{IV} &\le C\left(\int_{2^{k-2}r_B}^\infty\int_{|x-y|<t}\left|t^{-n}\sum_{k=1}^\infty|b_{2^{k+1}B}-b_B|\cdot\int_{2^{k+1}B\backslash 2^kB}|f(z)|\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &\le C\|b\|_*\left(\sum_{k=1}^\infty(k+1)\cdot\int_{2^{k+1}B\backslash 2^kB}|f(z)|\,dz\right)\left(\int_{2^{k-2}r_B}^\infty\frac{dt}{t^{2n+1}}\right)^{1/2}\\ &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{k=1}^\infty(k+1)\cdot w\big(2^{k+1}B\big)^{(\kappa-1)/p}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{w(B)^{\kappa/p}}\left(\int_B \mathrm{IV}^p\,w(x)\,dx\right)^{1/p} &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{k=1}^\infty(k+1)\cdot\frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}}\\ &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{k=1}^\infty\frac{k}{2^{kn\theta}}\\ &\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}, \qquad (11)\end{aligned}$$

where we have used the previous estimate (3) with $w \in RH_r$ and $\theta = (1-\kappa)(r-1)/pr$. Summarizing the estimates (9) and (11) derived above, we thus obtain

$$\frac{1}{w(B)^{\kappa/p}}\left(\int_B \mathrm{II}^p\, w(x)\,dx\right)^{1/p} \le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}. \tag{12}$$

Combining the inequalities (4), (6) with the above inequality (12) and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we complete the proof of Theorem 1.2. $\square$

4. Proofs of Theorems 1.3 and 1.4

In order to prove the main theorems of this section, we need to establish the following three lemmas.

Lemma 4.1. *Let $0 < \alpha \le 1$ and $w \in A_p$ with $p = 2$. Then for any $j \in \mathbb{Z}_+$, we have*

$$\left\|S_{\alpha,2^j}(f)\right\|_{L^2_w} \le C \cdot 2^{jn}\left\|S_\alpha(f)\right\|_{L^2_w}.$$

Proof. Since $w \in A_2$, then by Lemma 2.1, we get

$$w\big(B(y,2^jt)\big) = w\big(2^jB(y,t)\big) \le C\cdot 2^{2jn}w\big(B(y,t)\big) \quad j = 1,2,\ldots.$$

Therefore

$$\begin{aligned}
\left\|S_{\alpha,2^j}(f)\right\|^2_{L^2_w} &= \int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{n+1}_+}\Big(A_\alpha(f)(y,t)\Big)^2\chi_{|x-y|<2^jt}\frac{dydt}{t^{n+1}}\right)w(x)\,dx \\
&= \iint_{\mathbb{R}^{n+1}_+}\left(\int_{|x-y|<2^jt} w(x)\,dx\right)\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}} \\
&\le C\cdot 2^{2jn}\iint_{\mathbb{R}^{n+1}_+}\left(\int_{|x-y|<t} w(x)\,dx\right)\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}} \\
&= C\cdot 2^{2jn}\left\|S_\alpha(f)\right\|^2_{L^2_w}.
\end{aligned}$$

This finishes the proof of Lemma 4.1. $\square$

Lemma 4.2. *Let $0 < \alpha \le 1$, $2 < p < \infty$ and $w \in A_p$. Then for any $j \in \mathbb{Z}_+$, we have*

$$\left\|S_{\alpha,2^j}(f)\right\|_{L^p_w} \le C\cdot 2^{jnp/2}\left\|S_\alpha(f)\right\|_{L^p_w}.$$

Proof. For any $j \in \mathbb{Z}_+$, it is easy to see that

$$\left\|S_{\alpha,2^j}(f)\right\|^2_{L^p_w} = \left\|S_{\alpha,2^j}(f)^2\right\|_{L^{p/2}_w}.$$

Since $p/2 > 1$, then we have

$$\begin{aligned}
\left\|S_{\alpha,2^j}(f)^2\right\|_{L^{p/2}_w} &= \sup_{\|g\|_{L^{(p/2)'}_w}\le 1}\left|\int_{\mathbb{R}^n} S_{\alpha,2^j}(f)(x)^2 g(x)w(x)\,dx\right| \\
&= \sup_{\|g\|_{L^{(p/2)'}_w}\le 1}\left|\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{n+1}_+}\Big(A_\alpha(f)(y,t)\Big)^2\chi_{|x-y|<2^jt}\frac{dydt}{t^{n+1}}\right)g(x)w(x)\,dx\right| \\
&= \sup_{\|g\|_{L^{(p/2)'}_w}\le 1}\left|\iint_{\mathbb{R}^{n+1}_+}\left(\int_{|x-y|<2^jt} g(x)w(x)\,dx\right)\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\right|.
\end{aligned} \tag{13}$$

For $w \in A_p$, we denote the weighted maximal operator by M_w; that is

$$M_w(f)(x) = \sup_{x\in B}\frac{1}{w(B)}\int_B |f(y)|w(y)\,dy,$$

where the supremum is taken over all balls B which contain x. Then, by Lemma 2.1, we can get

$$\begin{aligned}\int_{|x-y|<2^j t} g(x)w(x)\,dx &\le C\cdot 2^{jnp} w\big(B(y,t)\big)\cdot \frac{1}{w(B(y,2^j t))}\int_{B(y,2^j t)} g(x)w(x)\,dx\\ &\le C\cdot 2^{jnp} w\big(B(y,t)\big)\inf_{x\in B(y,t)} M_w(g)(x)\\ &\le C\cdot 2^{jnp}\int_{|x-y|<t} M_w(g)(x)w(x)\,dx. \qquad (14)\end{aligned}$$

Substituting the above inequality (14) into (13) and using Hölder's inequality and the $L_w^{(p/2)'}$ boundedness of M_w, we thus obtain

$$\begin{aligned}\left\|S_{\alpha,2^j}(f)^2\right\|_{L_w^{p/2}} &\le C\cdot 2^{jnp}\sup_{\|g\|_{L_w^{(p/2)'}}\le 1}\left|\int_{\mathbb{R}^n} S_\alpha(f)(x)^2 M_w(g)(x)w(x)\,dx\right|\\ &\le C\cdot 2^{jnp}\left\|S_\alpha(f)^2\right\|_{L_w^{p/2}}\sup_{\|g\|_{L_w^{(p/2)'}}\le 1}\|M_w(g)\|_{L_w^{(p/2)'}}\\ &\le C\cdot 2^{jnp}\left\|S_\alpha(f)^2\right\|_{L_w^{p/2}}\\ &= C\cdot 2^{jnp}\|S_\alpha(f)\|^2_{L_w^p}.\end{aligned}$$

This implies the desired result. □

Lemma 4.3. *Let $0<\alpha\le 1$, $1<p<2$ and $w\in A_p$. Then for any $j\in\mathbb{Z}_+$, we have*

$$\left\|S_{\alpha,2^j}(f)\right\|_{L_w^p}\le C\cdot 2^{jn}\|S_\alpha(f)\|_{L_w^p}.$$

Proof. We will adopt the same method as in [26, pp. 315–316]. For any $j\in\mathbb{Z}_+$, set $\Omega_\lambda=\big\{x\in\mathbb{R}^n : S_\alpha(f)(x)>\lambda\big\}$ and $\Omega_{\lambda,j}=\big\{x\in\mathbb{R}^n : S_{\alpha,2^j}(f)(x)>\lambda\big\}$. We also set

$$\Omega_\lambda^*=\left\{x\in\mathbb{R}^n : M_w(\chi_{\Omega_\lambda})(x)>\frac{1}{2^{(jnp+1)}\cdot[w]_{A_p}}\right\}.$$

Observe that $w\big(\Omega_{\lambda,j}\big)\le w\big(\Omega_\lambda^*\big)+w\big(\Omega_{\lambda,j}\cap(\mathbb{R}^n\setminus\Omega_\lambda^*)\big)$. Thus

$$\begin{aligned}\left\|S_{\alpha,2^j}(f)\right\|^p_{L_w^p} &= \int_0^\infty p\lambda^{p-1}w\big(\Omega_{\lambda,j}\big)\,d\lambda\\ &\le \int_0^\infty p\lambda^{p-1}w\big(\Omega_\lambda^*\big)\,d\lambda+\int_0^\infty p\lambda^{p-1}w\big(\Omega_{\lambda,j}\cap(\mathbb{R}^n\setminus\Omega_\lambda^*)\big)\,d\lambda\\ &= \mathrm{I}+\mathrm{II}.\end{aligned}$$

The weighted weak type estimate of M_w yields

$$\mathrm{I}\le C\cdot 2^{jnp}\int_0^\infty p\lambda^{p-1}w(\Omega_\lambda)\,d\lambda = C\cdot 2^{jnp}\|S_\alpha(f)\|^p_{L_w^p}. \qquad (15)$$

To estimate II, we now claim that the following inequality holds.

$$\int_{\mathbb{R}^n\setminus\Omega_\lambda^*} S_{\alpha,2^j}(f)(x)^2 w(x)\,dx\le C\cdot 2^{jnp}\int_{\mathbb{R}^n\setminus\Omega_\lambda} S_\alpha(f)(x)^2 w(x)\,dx. \qquad (16)$$

We will take the above inequality temporarily for granted, then it follows from Chebyshev's inequality and (16) that

$$\begin{aligned}w\big(\Omega_{\lambda,j}\cap(\mathbb{R}^n\setminus\Omega_\lambda^*)\big) &\le \lambda^{-2}\int_{\Omega_{\lambda,j}\cap(\mathbb{R}^n\setminus\Omega_\lambda^*)} S_{\alpha,2^j}(f)(x)^2 w(x)\,dx\\ &\le \lambda^{-2}\int_{\mathbb{R}^n\setminus\Omega_\lambda^*} S_{\alpha,2^j}(f)(x)^2 w(x)\,dx\\ &\le C\cdot 2^{jnp}\lambda^{-2}\int_{\mathbb{R}^n\setminus\Omega_\lambda} S_\alpha(f)(x)^2 w(x)\,dx.\end{aligned}$$

Hence

$$\mathrm{II} \le C \cdot 2^{jnp} \int_0^\infty p\lambda^{p-1} \left(\lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_\alpha(f)(x)^2 w(x)\,dx \right) d\lambda.$$

Changing the order of integration yields

$$\begin{aligned} \mathrm{II} &\le C \cdot 2^{jnp} \int_{\mathbb{R}^n} S_\alpha(f)(x)^2 \left(\int_{|S_\alpha(f)(x)|}^\infty p\lambda^{p-3}\,d\lambda \right) w(x)\,dx \\ &\le C \cdot 2^{jnp} \frac{p}{2-p} \cdot \|S_\alpha(f)\|_{L^p_w}^p. \end{aligned} \tag{17}$$

Combining the above estimate (17) with (15) and taking p-th root on both sides, we complete the proof of Lemma 4.3. So it remains to prove the inequality (16). Set $\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega^*_\lambda) = \bigcup_{x \in \mathbb{R}^n \setminus \Omega^*_\lambda} \Gamma_{2^j}(x)$ and $\Gamma(\mathbb{R}^n \setminus \Omega_\lambda) = \bigcup_{x \in \mathbb{R}^n \setminus \Omega_\lambda} \Gamma(x)$. For each given $(y,t) \in \Gamma_{2^j}(\mathbb{R}^n \setminus \Omega^*_\lambda)$, by Lemma 2.1, we thus have

$$w\big(B(y,2^j t) \cap (\mathbb{R}^n \setminus \Omega^*_\lambda)\big) \le C \cdot 2^{jnp} w\big(B(y,t)\big).$$

It is not difficult to check that $w\big(B(y,t)\cap\Omega_\lambda\big) \le \frac{w(B(y,t))}{2}$ and $\Gamma_{2^j}(\mathbb{R}^n\setminus\Omega^*_\lambda) \subseteq \Gamma(\mathbb{R}^n\setminus\Omega_\lambda)$. In fact, for any $(y,t) \in \Gamma_{2^j}(\mathbb{R}^n\setminus\Omega^*_\lambda)$, there exists a point $x \in \mathbb{R}^n \setminus \Omega^*_\lambda$ such that $(y,t) \in \Gamma_{2^j}(x)$. Then we can deduce

$$\begin{aligned} w\big(B(y,t) \cap \Omega_\lambda\big) &\le w\big(B(y,2^j t) \cap \Omega_\lambda\big) \\ &= \int_{B(y,2^j t)} \chi_{\Omega_\lambda}(z) w(z)\,dz \\ &\le [w]_{A_p} \cdot 2^{jnp} w\big(B(y,t)\big) \cdot \frac{1}{w(B(y,2^j t))} \int_{B(y,2^j t)} \chi_{\Omega_\lambda}(z) w(z)\,dz. \end{aligned}$$

Note that $x \in B(y,2^j t) \cap (\mathbb{R}^n \setminus \Omega^*_\lambda)$. So we have

$$w\big(B(y,t) \cap \Omega_\lambda\big) \le [w]_{A_p} \cdot 2^{jnp} w\big(B(y,t)\big) M_w(\chi_{\Omega_\lambda})(x) \le \frac{w(B(y,t))}{2}.$$

Hence

$$\begin{aligned} w\big(B(y,t)\big) &= w\big(B(y,t) \cap \Omega_\lambda\big) + w\big(B(y,t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)\big) \\ &\le \frac{w(B(y,t))}{2} + w\big(B(y,t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)\big), \end{aligned}$$

which is equivalent to

$$w\big(B(y,t)\big) \le 2 \cdot w\big(B(y,t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)\big).$$

The above inequality implies in particular that there is a point $z \in B(y,t)\cap(\mathbb{R}^n\setminus\Omega_\lambda) \neq \emptyset$. In this case, we have $(y,t) \in \Gamma(z)$ with $z \in \mathbb{R}^n \setminus \Omega_\lambda$, which yields $\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega^*_\lambda) \subseteq \Gamma(\mathbb{R}^n \setminus \Omega_\lambda)$. Thus we obtain

$$w\big(B(y,2^j t) \cap (\mathbb{R}^n \setminus \Omega^*_\lambda)\big) \le C \cdot 2^{jnp} w\big(B(y,t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)\big).$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus \Omega^*_\lambda} S_{\alpha,2^j}(f)(x)^2 w(x)\,dx \\ &= \int_{\mathbb{R}^n \setminus \Omega^*_\lambda} \left(\iint_{\Gamma_{2^j}(x)} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \right) w(x)\,dx \\ &\le \iint_{\Gamma_{2^j}(\mathbb{R}^n\setminus\Omega^*_\lambda)} \left(\int_{B(y,2^j t)\cap(\mathbb{R}^n\setminus\Omega^*_\lambda)} w(x)\,dx \right) \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \\ &\le C \cdot 2^{jnp} \iint_{\Gamma(\mathbb{R}^n\setminus\Omega_\lambda)} \left(\int_{B(y,t)\cap(\mathbb{R}^n\setminus\Omega_\lambda)} w(x)\,dx \right) \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \\ &\le C \cdot 2^{jnp} \int_{\mathbb{R}^n\setminus\Omega_\lambda} S_\alpha(f)(x)^2 w(x)\,dx, \end{aligned}$$

which is just our desired conclusion. □

We are now in a position to give the proof of Theorem 1.3.

Proof of Theorem 1.3. From the definition of $g^*_{\lambda,\alpha}$, we readily see that

$$\begin{aligned} g^*_{\lambda,\alpha}(f)(x)^2 &= \iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \\ &= \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \\ &\quad + \sum_{j=1}^\infty \int_0^\infty \int_{2^{j-1}t\le|x-y|<2^jt} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \\ &\le C\left[\mathcal{S}_\alpha(f)(x)^2 + \sum_{j=1}^\infty 2^{-j\lambda n}\mathcal{S}_{\alpha,2^j}(f)(x)^2\right]. \end{aligned}$$

For any given ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$, then from the above inequality, it follows that

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}}\left(\int_B |g^*_{\lambda,\alpha}(f)(x)|^p w(x)\,dx\right)^{1/p} \\ &\le \frac{1}{w(B)^{\kappa/p}}\left(\int_B |\mathcal{S}_\alpha(f)(x)|^p w(x)\,dx\right)^{1/p} + \sum_{j=1}^\infty 2^{-j\lambda n/2}\cdot\frac{1}{w(B)^{\kappa/p}}\left(\int_B |\mathcal{S}_{\alpha,2^j}(f)(x)|^p w(x)\,dx\right)^{1/p} \\ &= I_0 + \sum_{j=1}^\infty 2^{-j\lambda n/2} I_j. \end{aligned}$$

By Theorem 1.1, we know that $I_0 \le C\|f\|_{L^{p,\kappa}(w)}$. Below we shall give the estimates of I_j for $j = 1, 2, \ldots$. As before, we set $f = f_1 + f_2, f_1 = f\chi_{2B}$ and write

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}}\left(\int_B |\mathcal{S}_{\alpha,2^j}(f)(x)|^p w(x)\,dx\right)^{1/p} \\ &\le \frac{1}{w(B)^{\kappa/p}}\left(\int_B |\mathcal{S}_{\alpha,2^j}(f_1)(x)|^p w(x)\,dx\right)^{1/p} + \frac{1}{w(B)^{\kappa/p}}\left(\int_B |\mathcal{S}_{\alpha,2^j}(f_2)(x)|^p w(x)\,dx\right)^{1/p} \\ &= I_j^{(1)} + I_j^{(2)}. \end{aligned}$$

Applying Lemmas 4.1–4.3, Theorem A and Lemma 2.1, we obtain

$$\begin{aligned} I_j^{(1)} &\le \frac{1}{w(B)^{\kappa/p}}\left\|\mathcal{S}_{\alpha,2^j}(f_1)\right\|_{L^p_w} \\ &\le C\Big(2^{jn} + 2^{jnp/2}\Big)\frac{1}{w(B)^{\kappa/p}}\cdot\|f_1\|_{L^p_w} \\ &\le C\|f\|_{L^{p,\kappa}(w)}\Big(2^{jn} + 2^{jnp/2}\Big)\cdot\frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\ &\le C\|f\|_{L^{p,\kappa}(w)}\Big(2^{jn} + 2^{jnp/2}\Big). \end{aligned}$$

We now turn to estimate the term $I_j^{(2)}$. For any $x \in B$, $(y,t) \in \Gamma_{2^j}(x)$ and $z \in \big(2^{k+1}B \setminus 2^kB\big) \cap B(y,t)$, then by a direct calculation, we can easily deduce

$$t + 2^jt \ge |x-y| + |y-z| \ge |x-z| \ge |z-x_0| - |x-x_0| \ge 2^{k-1}r_B.$$

Thus, it follows from the previous estimates (1) and (2) that

$$\begin{aligned} \left|\mathcal{S}_{\alpha,2^j}(f_2)(x)\right| &= \left(\iint_{\Gamma_{2^j}(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |f_2 * \varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\le C\left(\int_{2^{(k-2-j)}r_B}^\infty \int_{|x-y|<2^jt} \left|t^{-n}\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} |f(z)|\,dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \end{aligned}$$

$$\leq C\left(\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|f(z)|\,dz\right)\left(\int_{2^{(k-2-j)}r_B}^{\infty}2^{jn}\frac{dt}{t^{2n+1}}\right)^{1/2}$$
$$\leq C\cdot 2^{3jn/2}\sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B\setminus 2^kB}|f(z)|\,dz$$
$$\leq C\|f\|_{L^{p,\kappa}(w)}\cdot 2^{3jn/2}\sum_{k=1}^{\infty}w\left(2^{k+1}B\right)^{(\kappa-1)/p}.$$

Furthermore, by using (3) again, we get

$$I_j^{(2)}\leq C\|f\|_{L^{p,\kappa}(w)}\cdot 2^{3jn/2}\sum_{k=1}^{\infty}\frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}}$$
$$\leq C\|f\|_{L^{p,\kappa}(w)}\cdot 2^{3jn/2}.$$

Therefore

$$\frac{1}{w(B)^{\kappa/p}}\left(\int_B\left|g^*_{\lambda,\alpha}(f)(x)\right|^p w(x)\,dx\right)^{1/p}\leq C\|f\|_{L^{p,\kappa}(w)}\left(1+\sum_{j=1}^{\infty}2^{-j\lambda n/2}2^{3jn/2}+\sum_{j=1}^{\infty}2^{-j\lambda n/2}2^{jnp/2}\right)$$
$$\leq C\|f\|_{L^{p,\kappa}(w)},$$

where the last two series are both convergent under our assumption $\lambda > \max\{p, 3\}$. Hence, by taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we conclude the proof of Theorem 1.3. □

Finally, we remark that by using the arguments as in the proofs of Theorems 1.2 and 1.3, we can also show the conclusion of Theorem 1.4. The details are omitted here.

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